MA135—VECTORS AND MATRICES
Example Sheet 5

All questions in Sections A and B must be handed in to your supervisor via the supervisor's pigeon loft by **3pm Monday, Week 6.** Section C questions should be **attempted** by students who hope to get a 1st or 2:1 degree, but are not to be handed in.

Section A

A1 Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Explain why $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$ is undefined and why $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is defined.

A2 (i) Compute $(\mathbf{i}\times\mathbf{i})\times\mathbf{j}$ and $\mathbf{i}\times(\mathbf{i}\times\mathbf{j})$. What do you notice? What is wrong with writing $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$?
(ii) Let $\mathbf{u}=(1,0,3)$, $\mathbf{v}=(0,1,-1)$. Compute $\mathbf{u}\times\mathbf{v}$. Hence find the two unit vectors that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
(iii) Find the area of the parallelogram that has **unit** vectors $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides, if $\mathbf{u}\cdot\mathbf{v}=\sqrt{3}/2$.

A3 Let $A=\operatorname{diag}(\alpha_1,\dots,\alpha_n)$. Use the definition of eigenvalue to show that $\alpha_1,\dots,\alpha_n$ are eigenvalues of A (this means that you must find a vectors $\mathbf{v}_1,\dots,\mathbf{v}_n$ such that $A\mathbf{v}_i=\alpha_i\mathbf{v}_i$).

A4 Suppose A is a square matrix and λ is an eigenvalue of A.
(i) Show that λ^n is an eigenvalue of A^n for all positive integers n.
(i) Suppose A is invertible. Show λ is non-zero and that λ^{-1} is and eigenvalue of A^{-1}.

A5 Let A, B be $n\times n$ matrices. Suppose that $\mathbf{v}$ is an eigenvector to both A and B. Show that $\mathbf{v}$ is an eigenvector to AB and to $A+B$.

A6 Let $A=2I_n$, $B=-I_n$ and $C=I_n$. What do the linear transformations T_A, T_B and T_C represent geometrically?

A7 Use the triple product to evaluate the determinant matrix, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 0 & 0 \end{pmatrix}$.

Section B

B1 Find the two unit vectors parallel to the xy-plane that are perpendicular to the vector $(-2,3,5)$.

B2 Let $A=\left(\begin{smallmatrix} 0 & -1 \\ 2 & 3 \end{smallmatrix}\right)$.
(i) Calculate the eigenvalues of A and corresponding eigenvectors.
(ii) Give a matrix P the diagonalizes A.
(iii) Calculate A^n for positive n.

B3 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. This is projection from 3-space onto the xy-plane. Show that T is a linear transformation. What is the matrix associated to T?

B4 Let A be a 2×2 matrix. The characteristic polynomial of A is defined to be $\chi_A(x) = \det(xI_2 - A)$ (you know that the eigenvalues are the roots of this). Now suppose that A and B are similar matrices.
(i) Show that A, B have the same characteristic polynomial (**Hint:** write $B = P^{-1}AP$ and $I_2 = P^{-1}I_2P$ in the definition of $\chi_B(x)$ and show that this is equal to $\chi_A(x)$.)
(ii) Part (i) shows that similar matrices have the same eigenvalues. Can you show this directly from the definition of eigenvalue?

Section C

C1 Suppose A, B, C are $n \times n$ matrices. Prove the following.
(i) A is similar to A.
(ii) If A is similar to B then B is similar to A.
(iii) If A is similar to B and B is similar to C then A is similar to C.

C2 Let $R_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$.
(i) What is $\det(R_\theta)$?
(ii) Show that R_θ is orthogonal (recall that an $n \times n$ matrix A is **orthogonal** if $A^tA = AA^t = I_n$).
(iii) Show algebraically that $R_\phi R_\theta = R_{\phi+\theta}$.
(iv) Use the geometric interpretation of the matrix R_θ to explain (iii).
(v) Use the geometric interpretation of the matrix $R_{\pi/2}$ to explain why it cannot have real eigenvalues and eigenvectors.
(vi) Compute the eigenvalues and corresponding eigenvectors for $R_{\pi/2}$. Hence diagonalize $R_{\pi/2}$.

C3 Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an orthogonal matrix. We will show in steps that $A = \pm R_\theta$ for some θ.
(i) Show that $a^2 + c^2 = 1$, $b^2 + d^2 = 1$ and $ab + cd = 0$.
(ii) Deduce the existence of angles ϕ, ψ such that $a = \cos\phi$, $c = \sin\phi$, $b = \cos\psi$, $d = \sin\psi$.
(iii) Substitute into $ab + cd = 0$ and deduce that $\phi = \psi \pm \pi/2$.
(iv) Deduce that $A = \pm R_\theta$ for some θ.
(v) You know that R_θ represents anti-clockwise rotation about the origin through an angle θ. Describe in words the linear transformation associated with the matrix $-R_\theta$ (**Warning: don't be rash!**).

C4 Let $\mathbf{w} \in \mathbb{R}^3$. Show that the map $S_\mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S_\mathbf{w}(\mathbf{u}) = \mathbf{w} \times \mathbf{u}$ is a linear transformation (**Hint:** use the properties of the vector product). What is the matrix associated with $S_\mathbf{i}$? Describe the image of $S_\mathbf{i}$.

C5 Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that A is not diagonalizable (**Hint:** Use proof by contradiction). Why does this not contradict the theorem we took in the lectures about diagonalizing matrices?

C6 Give an explicit linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the plane $x+y+z = 0$ (**Hint:** It would help to write the equation of the plane in vector form).