MA3D5 Galois Theory

Homework Assignment 4

The deadline is **2pm Thursday, week 9.** Please hand in your solutions to Questions **1 and 2** to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Let $L = \mathbb{Q}(\zeta, \sqrt[3]{2})$ where $\zeta = \exp(2\pi i/3)$. In the lectures we showed that $L/\mathbb{Q}$ is Galois and identified its Galois group with S_3, by noting that L is the splitting field of $f = x^3 - 2$, and ordering the roots of f as $\sqrt[3]{2}$, $\zeta\sqrt[3]{2}$, $\zeta^2\sqrt[3]{2}$.
   (a) Give the following as subgroups of S_3:
   $$\mathbb{Q}(\sqrt[3]{2})^*, \qquad \mathbb{Q}(\zeta)^*.$$
   (b) Calculate the following intermediate fields
   $$\{1,\ (1,2,3),\ (1,3,2)\}^\dagger, \qquad \{1,\ (2,3)\}^\dagger.$$
   (c) With the help of the Fundamental Theorem of Galois Theory show that there are precisely six intermediate fields F for the extension $L/\mathbb{Q}$ (including L, $\mathbb{Q}$), and identify the ones for which $F/\mathbb{Q}$ is Galois.

2. Let $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3})$.
   (a) Show that $[L : \mathbb{Q}] = 8$. (Hint: you may use the fact that $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4$ for distinct primes p, q.)
   (b) Show that $L/\mathbb{Q}$ is Galois, and compute its Galois group as a subgroup of S_6, by noting that L is the splitting field of $f = (x^2+1)(x^2-2)(x^2-3)$ and ordering the roots of f as i, $-i$, $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{3}$, $-\sqrt{3}$.
   (c) Give the following as subgroups of S_6:
   $$\mathbb{Q}^*, \qquad \mathbb{Q}(\sqrt{-1})^*, \qquad \mathbb{Q}(\sqrt{2}, \sqrt{3})^*, \qquad L^*.$$
   (d) Calculate the following intermediate fields:
   $$\{1,\ (3,4)\}^\dagger, \qquad \{1,\ (1,2)(5,6)\}^\dagger, \qquad \{1,\ (1,2),\ (5,6),\ (1,2)(5,6)\}^\dagger.$$
   (e) Explain why $F/\mathbb{Q}$ is Galois for all intermediate fields F of $L/\mathbb{Q}$.

3. Let f be a squarefree separable polynomial over K. Let $L = K(\alpha_1, \ldots, \alpha_n)$ be the splitting field of f where $\alpha_1, \ldots, \alpha_n$ are the roots of f. Define the **discriminant** of f to be
   $$D(f) = \left(\prod_{1 \le i < j \le n} (\alpha_i - \alpha_j) \right)^2.$$
   (i) Show that $D(f) \in K$.
   (ii) Show that $D(f)$ is a square in K if and only if $\mathrm{Aut}(L/K) \subseteq A_n$.
   Hint: Revise alternating polynomials in your Introduction to Abstract Algebra notes.

4. Let p be a prime and let K be a finite field having p^6 elements. With the help of the Fundamental Theorem of Galois Theory, determine the number of intermediate fields for $K/\mathbb{F}_p$. How many elements does each have?