

# MA3D5 Galois Theory

## Homework Assignment 1

The deadline is **2pm Thursday, week 3**. Please hand in your answers to questions **5, 6** to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Show that  $f$  is irreducible over the given field  $K$ :
  - (a)  $f = x^5 + 4x^2 - 6$  over  $\mathbb{Q}$ .
  - (b)  $f = x^5 + t^2x^2 - 3t$  over  $\mathbb{F}_5(t)$ .
  - (c)  $f = x^{p-1} + x^{p-2} + \cdots + 1$  over  $\mathbb{Q}$ , where  $p$  is a prime.
2. Let  $f = x^3 + x + 3$ .
  - (a) Show that  $f$  is irreducible over  $\mathbb{Q}$ .
  - (b) Show that  $f$  has exactly one real root.
3. Let  $p$  be a prime. Show in  $\mathbb{F}_p[x, y]$  that
$$(x + y)^p = x^p + y^p.$$
4. Let  $\mathfrak{a} = (1 + i)\mathbb{Z}[i]$ . Show that  $\mathbb{Z}[i]/\mathfrak{a}$  is a field. How many elements does it have? Write down addition and multiplication tables for the elements.
5. Let  $p, q$  be distinct primes. Show that  $\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ .
6. Compute and simplify the splitting fields of  $f \in K[x]$  over the given  $K$ .
  - (a)  $f = (x^2 + x + 1)(x^2 - 5)$ ,  $K = \mathbb{Q}$ .
  - (b)  $f = (x^2 + x - 1)(x^2 - 5)$ ,  $K = \mathbb{Q}$ .
  - (c)  $f = x^3 - 7$ ,  $K = \mathbb{Q}$ .
  - (d)  $f = x^3 - 7$ ,  $K = \mathbb{Q}(\sqrt{-3})$ .
7.
  - (a) Let  $f$  be an irreducible quadratic polynomial over  $\mathbb{Q}$ . Show that its splitting field has the form  $\mathbb{Q}(\sqrt{D})$  where  $D$  is a squarefree integer  $\neq 0, 1$ .
  - (b) Let  $f = x^3 - 3x + 1$ . Show that its splitting field over  $\mathbb{Q}$  is contained in  $\mathbb{R}$ .
8. **Very hard! Don't spend too much time on this.** Show that  $x^n + x + 3$  is irreducible for all  $n \geq 2$ .
9. **Aptitude test for prospective university administrators** Reformulate the above questions and your answers in the new Warwick tone of voice.