MA136 Introduction to Abstract Algebra

Homework Assignment 3

Please Enjoy Responsibly!

Attempt all the questions on this sheet, and hand in solutions to A2, A4, A5, B1, B2, B3, B4. Solutions to your supervisor's pigeon-loft by **2pm Thursday, week 9.** Your supervisor will mark some, but not all, of these questions. They don't know which ones yet, so there's no point asking them.

(A1) Show that cyclic groups are abelian.

(A2) In each of the following groups G, write down the cyclic subgroup generated by g.
(a) $G = \mathbb{S}$, $g = \exp(2\pi i/7)$.
(b) $G = \mathbb{Z}/12\mathbb{Z}$, $g = \overline{8}$.
(c) $G = \mathrm{GL}_2(\mathbb{R})$, $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
(d) $G = \mathbb{R}/\mathbb{Z}$, $g = \overline{5/7}$.
(e) $G = D_4$, $g = \rho_3$.

(A3) Which of the following groups G are cyclic? Justify your answer for each, and if G is cyclic then write down a generator.
(a) $G = k\mathbb{Z}$ (where k is a non-zero integer).
(b) $G = \mathbb{Z}/m\mathbb{Z}$ (where m is a positive integer).
(c) D_3.

(A4) For the following groups G and subgroups H, write down the (left) cosets of H in G and determine the index $[G : H]$.
(a) $G = 2\mathbb{Z}$ and $H = 6\mathbb{Z}$.
(b) $G = U_4$ and $H = U_2$.
(c) $G = D_4$ and $H = \langle \rho_1 \rangle$.
(d) $G = \mathbb{R}^*$ and $H = \{a \in \mathbb{R} : a > 0\}$.

(A5) There are four elements of order 5 in $\mathbb{R}/\mathbb{Z}$; find them. There are eight elements of order 3 in $\mathbb{R}^2/\mathbb{Z}^2$; find them.

(A6) In $\mathbb{Z}^2$ we let $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ as usual. Write $2\mathbb{Z}^2 = \{(2a, 2b) : a, b \in \mathbb{Z}\}$. Convince yourself that $2\mathbb{Z}^2$ is a subgroup of $\mathbb{Z}^2$ of index 4 and that

$$\mathbb{Z}^2/2\mathbb{Z}^2 = \{\overline{\mathbf{0}}, \overline{\mathbf{i}}, \overline{\mathbf{j}}, \overline{\mathbf{i}+\mathbf{j}}\}.$$

Write down an addition table for $\mathbb{Z}^2/2\mathbb{Z}^2$.

(B1) In this exercise, you will show using contradiction that $\mathbb{R}^*$ is not cyclic. Suppose that it is cyclic and let $g \in \mathbb{R}^*$ be a generator. Then $\mathbb{R}^* = \langle g \rangle$. In particular, $|g|^{1/2} \in \mathbb{R}^*$ and so $|g|^{1/2} = g^m$ for some integer m. Show that the only solutions to this equation are $g = \pm 1$. Where's the contradiction?

(B2) Let $\beta \in \mathbb{C}^*$ and write $\beta = re^{i\phi}$ where r, $\phi \in \mathbb{R}$ and $r > 0$. Show that $\beta\mathbb{S} = r\mathbb{S}$. What does the coset $r\mathbb{S}$ represent geometrically?

(B3) Let G be a group of order p, where p is a prime number. Let H be a subgroup. Show that H must either equal G or the trivial subgroup $\{1\}$ (**Hint: Use Lagrange's Theorem**). Deduce that if $g \in G$ is not the identity element, then $G = \langle g \rangle$.

(B4) Let $\alpha \in [0, 1)$. Show that $\overline{\alpha} \in \mathbb{R}/\mathbb{Z}$ has finite order if and only if α is rational.

(C1) Show that every non-zero element of $\mathbb{R}/\mathbb{Q}$ has infinite order.

(C2) Let $\mathbf{v}$ be a column vector in $\mathbb{R}^2$ and H a subgroup of $\mathrm{GL}_2(\mathbb{R})$. We define the orbit of $\mathbf{v}$ under H to be the set

$$\mathrm{Orb}(H, \mathbf{v}) = \{A\mathbf{v} : A \in H\}.$$

Observe $\mathrm{Orb}(H, \mathbf{0}) = \{\mathbf{0}\}$. Suppose $\mathbf{v} \neq \mathbf{0}$.

(i) Let $H = \{\lambda I_2 : \lambda \in \mathbb{R}^*\}$. Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Describe and sketch $\mathrm{Orb}(H, \mathbf{v})$.

(ii) Let $H = \mathrm{SO}_2(\mathbb{R})$ (this is the subgroup of rotation matrices—see Sections IV.7 and IX.4 of the lecture notes). Describe and sketch $\mathrm{Orb}(H, \mathbf{v})$.

(iii) Finally, let $H = \mathrm{GL}_2(\mathbb{R})$. With the help of (i) and (ii) explain why $\mathrm{Orb}(H, \mathbf{v}) = \mathbb{R}^2 \backslash \{\mathbf{0}\}$.

(C3) Let $\mathbf{v}$ be a column vector in $\mathbb{R}^2$. In the previous assignment we defined the stabilizer of $\mathbf{v}$ to be

$$\mathrm{Stab}(\mathbf{v}) = \{A \in \mathrm{GL}_2(\mathbb{R}) : A\mathbf{v} = \mathbf{v}\},$$

and showed that it is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Now let $\mathbf{v}$ and $\mathbf{w}$ be non-zero column vectors in $\mathbb{R}^2$.

(i) Show that there is some $B \in \mathrm{GL}_2(\mathbb{R})$ such that $B\mathbf{v} = \mathbf{w}$. **Hint:** Use part (iii) of (C2).

(ii) Let $U = \{C \in \mathrm{GL}_2(\mathbb{R}) : C\mathbf{v} = \mathbf{w}\}$. With the help of (i), show that U is a left coset of $\mathrm{Stab}(\mathbf{v})$.

(iii) Show that $\mathrm{Stab}(\mathbf{v}) = \{B^{-1}AB : A \in \mathrm{Stab}(\mathbf{w})\}$ where B is as in (i).