Beyond Fermat's Last Theorem

Provisional Aim Extend the proof of FLT so that we can solve other (famous) Diophantine problems.

Proof Sketch of FLT (Wiles)

Suppose $a, b, c \in \mathbb{Z}$ are coprime, $abc \neq 0$
$$a^p + b^p + c^p = 0, \quad p \geq 5 \text{ prime.}$$
Associate this to the 'Frey elliptic curve'

$$E: \quad Y^2 = X(X - a^p)(X + b^p)$$

Wiles: E is modular.

Ribet's Thm $\Longrightarrow$ Galois representation on $E[p]$ arises from a cusp form of level 2.

There are no cusp forms at level 2. Contradiction. $\square$

Aim To combine

1. Baker's Theory for bounding exponents and variables,
2. Modular Approach used in the proof of FLT,

to solve (famous) Diophantine equations.

Joint work with Y. Bugeaud, M. Mignotte } Strasbourg

Theorem (BMS) The only perfect powers in the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0)$$

are $F_0 = 0$, $F_1 = F_2 = 1$

$F_6 = 8$, $F_{12} = 144$.

Theorem (BMS) The only solutions to

$x^2 + 7 = y^m$ $(m \geq 3)$ are

m	3	3	4	5	5	7	15
x	± 1	± 181	± 3	± 5	± 181	± 11	± 181
y	2	32	± 2	2	8	2	2

Theorem (BMS) Suppose $3 \leq q < 100$ is prime. Then the only solutions to

$$q^u x^n - 2^v y^n = \pm 1 \qquad n \geq 3,\ xy \neq 0,\ u, v \geq 0$$

are

$1 - 2 = -1$, $3 - 2 = 1$, $3 - 4 = -1$,
$9 - 8 = 1$, $5 - 4 = 1$, $7 - 8 = -1$,
$17 - 16 = 1$, $31 - 32 = -1$,
$5 \times 2^4 - 3^4 = -1$, $19 \times 3^3 - 8^3 = 1$,
$17 \times 7^3 - 18^3 = -1$, $37 \times 3^3 - 10^3 = -1$,
$43 \times 2^3 - 7^3 = 1$, $53 - 2 \times 3^3 = -1$.

Proved using the multi-Frey approach + new bounds for linear forms in 3 logs.

Some snippets from the proof of the Fibonacci powers theorem.

Suppose $F_n = y^p$ $\quad p \geq 7$ prime, n odd (hard case)

Let $\lambda = \frac{1+\sqrt{5}}{2}$ $\quad \mu = \frac{1-\sqrt{5}}{2}$

Then $F_n = \frac{1}{\sqrt{5}}(\lambda^n - \mu^n)$

Define $\ell_n = \lambda^n + \mu^n$ (companion Lucas sequence)

Frey curve $E_n : Y^2 = X^3 + \ell_n X^2 - X$

Ribet's Thm $\Longrightarrow$ .... arises from a cusp form of level 20

There is a unique cusp form at level 20 corresponding to

$$E : Y^2 = X^3 + X^2 - X$$

'..... arises from ...' means:

for all primes $l \neq 2, 5$

(i) if $l \nmid F_n$ then
$(\# E_n \bmod l) \equiv (\# E \bmod l) \bmod p$

(ii) if $l \mid F_n$ then something similar.

Thus $\ell_n \equiv \alpha_1, \dots, \alpha_r \mod \ell$

(Recall $\ell_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$)

Choose $\ell \equiv \pm 1 \pmod 5$

$\Longrightarrow n \equiv \beta_1, \dots, \beta_s \mod \underbrace{(\ell - 1)}_{\text{composite}}$

Step I

$$\frac{\lambda^n - \mu^n}{\sqrt{5}} = F_n = y^p$$

$$\Longrightarrow \quad \frac{K}{y^{C \cdot \log p}} \leq \left| \frac{\lambda^n}{\sqrt{5}\, y^p} - 1 \right| = \frac{1}{\sqrt{5}\, \lambda^n y^p}$$

Baker's idea

Improving on Baker

$$p \leq 2 \times 10^8 \quad (\text{if } n > 1)$$

Step II For each $7 \leq p \leq 2 \times 10^8$

choose primes $\ell \equiv 1 \pmod p$ & $\equiv \pm 1 \pmod 5$

Then

$$n \equiv \beta_1, \dots, \beta_s \pmod p$$

Repeat until we have shown that

$$n \equiv \pm 1 \pmod p$$

Step III Shown $n = kp \pm 1$

$$\left|\left(\frac{\lambda^k}{y}\right)^p\left(\frac{\lambda^{\pm 1}}{\sqrt{5}}\right) - 1\right| = \left|\frac{\lambda^n}{\sqrt{5}\, y^p} - 1\right|$$
$$= \frac{1}{\sqrt{5}\,\lambda^n y^p}$$

$\Longrightarrow \quad p \leqslant 733$

Step IV $n \lesssim p^{10p}$

Step V Fix $7 \leqslant p \leqslant 733$

From step IV $n \leqslant B$

For $\ell \equiv \pm 1 \pmod 5$ we have

$$n \equiv \beta_1, \ldots, \beta_s \mod (\ell - 1)$$

Choose many ℓ such that $(\ell - 1)$ is 'very smooth'. Keep chinese remaindering until

$$n \equiv 1, a, b, c, \ldots \mod M$$

where $1 < a < b < c < \ldots < M$ and $B < a$.

$\therefore \quad n = 1$ end of proof.

papers

1. Bugeaud, Mignotte and Siksek
"Classical and Modular Approaches to Exponential Diophantine Equations
I. Fibonacci and Lucas Perfect Powers"
To appear in Annals of Math.

2. Bugeaud, Mignotte and Siksek
"Classical ... II. The Lebesgue-Nagell Equation"
To appear in Compositio Math.

3. Bugeaud, Mignotte and Siksek
"A Multi-Frey Approach to a Family of Diophantine Equations."

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