Giving Baker's Theory a Modular Helping Hand

- joint work with Cremona (Nottingham)
- joint work with Bugeaud & Mignotte (Strasbourg)

Baker's Theory: lower bounds for linear forms in logs

$\Rightarrow$ effective (astronomical) bounds to solutions of some equations

Baker + LLL: can solve
De Weger

1. Thue equations
2. S-unit equations
3. Integral points on curves

etc.

Baker + LLL can't solve

$$x^2 + 7 = y^p$$ (suggested by Cohn)

Baker & Wüsthulz $\overset{}{\underset{\text{Lesage}}{\Longrightarrow}}$ $p \le 6.6 \times 10^{15}$

Matveev $\Longrightarrow$ $p \le 6.81 \times 10^{12}$

<u>Cremona/Siksek</u> try modular approach (mimic proof of FLT)

Can assume $2 \mid y$, $p \ge 11$

Frey curve:

$$E_x : Y^2 = X^3 + xX^2 + \left(\frac{x^2+7}{4}\right)X$$

$$\Delta_{min} = \frac{-7}{2^{12}} y^{2p} \qquad N = 14 \prod_{\substack{\ell \mid y,\ \ell \ne 2,7}} \ell$$

Ribet's Level-Lowering Thm $\Longrightarrow$

Galois representation on $E_x[p]$ arises from a newform of level 14

$E: Y^2 + XY + Y = X^3 + 4X - 6$ 14A1

[Diverged from proof of FLT]

'arises from' $\Rightarrow$

(i) $a_\ell(E_x) \equiv a_\ell(E) \bmod p$ $\quad \ell \nmid 14y$

(ii) $\ell + 1 \equiv \pm a_\ell(E) \bmod p$ $\quad \ell \mid y,\ \ell \nmid 14$

Apply an idea of Kraus:

[solved $a^3 + b^3 = c^p$ $\quad 11 \leq p \leq 10^4$]

Fix p, choose prime ℓ s.t.

$$\ell - 1 = np \ \&\ p \nmid (\ell + 1 \pm a_\ell(E))$$

Let $\zeta_1, \dots, \zeta_n$ be the nth roots of unity in $\mathbb{F}_\ell^*$

If $x^2 + 7 = y^p$ then

$$x^2 + 7 \equiv \zeta_1, \dots, \zeta_n \bmod \ell$$

Solve for x $\Rightarrow$

$x \equiv x_1, \dots, x_t \mod \ell$

If $a_\ell(E_{x_i}) \not\equiv a_\ell(E) \mod p \ (\forall i)$

we get a contradiction.

Thm (Cremona / Siksek)

$$x^2 + 7 = y^p$$

has no solutions for $11 \leq p \leq 10^8$

Bugeaud, Mignotte & Siksek

Gave new lower bound for linear forms in 3 logs

For $x^2 + 7 = y^p$, $y \geq 22$

Baker & Wüstholz	$p \leq 6.6 \times 10^{15}$
Matveev	$p \leq 6.81 \times 10^{12}$
BMS	$p \leq 1.11 \times 10^{9}$

But

We can suppose $y \neq 2^r$

Let $\ell \mid y \qquad \ell \neq 2, 7$

$$\Rightarrow \ell + 1 \equiv \pm a_\ell(E) \mod p$$

$$\Rightarrow p \mid \ell + 1 \pm a_\ell(E)$$

$$\Rightarrow p \leq \ell + 1 + 2\sqrt{\ell}$$

$$\Rightarrow \ell \geq (\sqrt{p} - 1)^2$$

$$\Rightarrow \underbrace{y \geq (\sqrt{p} - 1)^2}_{\text{"Modular lower Bound for } y\text{"}} \geq 9999^2$$

$\left.\begin{array}{l}\text{BMS lower bound for 3 logs} \\ + \text{Modular lower bound for } y\end{array}\right\}$

$$\Rightarrow p \leq 1.81 \times 10^8$$

Thm Only solns to $x^2 + 7 = y^m$, $m \geq 3$

m	3	3	4	5	5	7	15
x	± 1	± 181	± 3	± 5	± 181	± 11	± 181
y	2	32	± 2	2	8	2	2

We also solved

$$x^2 + D = y^m, \quad m \geq 3, \qquad 1 \leq D \leq 100$$

using
(i) BMS Lower Bound for linear forms in 3 logs
(ii) modular lower bound for y
(iii) 3 modular methods
(iv) 206 days of computations on many machines.

Fibonacci Perfect Powers (BMS)

$$F_0 = 0, \quad F_1 = 1, \ \ldots \quad F_{n+2} = F_n + F_{n+1}$$

Conjecture (Cohn 1964) The only perfect powers in Fibonacci sequence are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_6 = 8$, $F_{12} = 144$.

i.e. solve $F_n = y^p$ (always has soln $(n, y, p) = (1, 1, p)$)

solved for $p = 2$ by Cohn & Wyler (indep) 1964

solved for $p = 3$ by London & Finkelstien 1969

$F_n = y^p \Longrightarrow$ Thue eqn of degree p

Can be solved by Baker + LLL

Resolved for $p=3$ Pethő (1983)

$p = 5, 7, 11, 13, 17$ McLaughlin (2000)

Pethő (indep) & Robbins 1983 } If $p \geq 3$, $n \geq 7$ & $F_n = y^p$ $\Longrightarrow \exists\, q \mid n$ s.t. $F_q = y_1^p$

Reduce to

$$F_n = y^p \qquad (n, p \text{ prime})$$

Recall $F_n = \dfrac{\omega^n - \bar{\omega}^n}{\sqrt{5}}$ $\qquad \omega = \dfrac{1+\sqrt{5}}{2}$

Let $x = \begin{cases} \omega^n + \bar{\omega}^n & n \equiv 1 \pmod 6 \\ -(\omega^n + \bar{\omega}^n) & n \equiv 5 \pmod 6 \end{cases}$

Then $x^2 + 4 = 5F_n^2$

$$\Longrightarrow \quad x^2 + 4 = 5y^{2p}$$

<u>Frey curve</u> $E_x : Y^2 = X^3 + xX^2 - X$

<u>Level lowering</u> $E : Y^2 = X^3 + X^2 - X$

$$\Longrightarrow (*) \begin{cases} a_\ell(E_x) \equiv a_\ell(E) \pmod{p} & \ell \nmid 10y \\ \ell + 1 \equiv \pm a_\ell(E) \pmod{p} & \ell \mid y,\ \ell \nmid 10 \end{cases}$$

<u>Example</u> $p = 7 \Longrightarrow n = 1$.

<u>Proof</u>

$p = 7 \Longrightarrow n < 2{\cdot}639 \times 10^{46}$

using a refinement of Bugeaud & Győry 1996 bounds for Thue equations.

Choose prime $\ell \neq 2, 5$ s.t. $\left(\frac{5}{\ell}\right) = 1$

$$(\ell - 1) \mid \underbrace{2^5 \times 3^3 \times 5^2 \times \cdots \times 109}_{M}$$

Write down $x \pmod{\ell}$ s.t. $(*)$ is satisfied

$$x \equiv x_1, \ldots, x_k \pmod{\ell}$$

$\Longrightarrow$ $n \equiv n_1, \ldots, n_t \mod \operatorname{lcm}(6, \ell - 1)$

Do this for many primes ℓ and Chinese-Remainder

Using about 130 primes l

$$\Longrightarrow \quad n \equiv 1,\ a,\ b,\ c \mod M$$

$$a \approx 1.007 \times 10^{47}, \quad b, c > a.$$
$$M \approx 2 \times 10^{47}$$

But $n \leq 2.639 \times 10^{46} \Longrightarrow n = 1.$

Using 6262 primes l we solve

$$F_n = y^p \quad \text{for} \quad 7 \leq p \leq 733$$

(for $p = 733$ we get $n \leq 10^{8733}$)

What bound can be proved for p?

$$F_n = \frac{\omega^n - \bar{\omega}^n}{\sqrt{5}} = y^p$$

$$\Longrightarrow \left| \frac{\omega^n}{\sqrt{5}\, y^p} - 1 \right| = \frac{1}{\sqrt{5}\, \omega^n\, y^p}$$

$$\Longrightarrow \frac{K}{y^{K' \log p}} \leq \left| n \log \omega - p \log y - \log \sqrt{5} \right| < \frac{C_1}{y^{2p}}$$

Baker's idea

BMS lower bound for linear forms in 3 logs $\Big\} \Longrightarrow p \leq 2 \times 10^8$

But using the modular approach we can prove that

$$n \equiv \pm 1 \pmod{p}$$

$$\Longrightarrow n = sp + \varepsilon \qquad \varepsilon = \pm 1$$

$$\Lambda = n \log \omega - p \log y - \log \sqrt{5}$$
$$= \underbrace{p \log\left(\frac{\omega^s}{y}\right) - \log\left(\sqrt{5}/\omega^{\varepsilon}\right)}_{\text{linear form in 2 logs!!}}$$

Laurent, Mignotte & Nesterenko lower bound for 2 logs $\Big\} \Longrightarrow$

$$p \leq 733$$

<u>Theorem</u> (BMS) The only perfect powers in Fibonacci sequence are

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_6 = 8,\ F_{12} = 144$

Only perfect powers in Lucas sequence are $L_1 = 1,\ L_3 = 4$

$(L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_n + L_{n+1})$