EXERCISES 2

Exercise 1. Using the recipes in Section 14.2, write down a Frey curve for

$$u^2 + 25 = v^{23}.$$

Use Kraus to show it does not have any solutions.

Exercise 2. In this exercise, you will solve

$$x^{2p} + y^{2p} = z^5, \qquad x,\ y,\ z \text{ coprime},\ p \text{ prime},\ p \geq 7.$$

(i) Show z is odd. **Without loss of generality x is even and y is odd.**

(ii) Show that

$$x^p + iy^p = (u + iv)^5$$

for some integers u, v.

(iii) Deduce that

$$x^p = u(u^4 - 10u^2v^2 + 5v^4), \qquad y^p = v(5u^4 - 10u^2v^2 + v^4).$$

(iv) Show that u, v are coprime, with u even.

(v) **Case I:** Suppose that $5 \nmid uv$.

- Show that

$$u = A^p, \qquad u^4 - 10u^2v^2 + 5v^4 = B^p,$$
$$v = C^p, \qquad 5u^4 - 10u^2v^2 + v^4 = D^p.$$

- Deduce

$$D^p + 20A^{4p} = w^2$$

for an appropriate integer w.

- Use an appropriate Frey curve to deduce a contradiction.

(vii) **Case II:** Repeat for $5 \mid uv$.

Exercise 3. Let p be a prime. Denote by ζ_p a primitive p-th root of unity. Define $\chi_p : G_{\mathbb{Q}} \to \mathbb{F}_p^*$ as follows: if $\sigma \in G_{\mathbb{Q}}$ then $\sigma(\zeta_p) = \zeta_p^{\chi_p(\sigma)}$. We call χ_p the **mod p cyclotomic character**.

(i) Show that χ_p is a character (i.e. a homomorphism).

(ii) Let $\sigma \in G_{\mathbb{Q}}$ denote complex conjugation. Show that $\chi_p(\sigma) = -1$.

(iii) Let $\ell \neq p$ be a prime, and let σ_ℓ be a Frobenius element at ℓ. Show that $\chi_p(\sigma_\ell) \equiv \ell \pmod{p}$.

(iv) Let $E/\mathbb{Q}$ an elliptic curve and let $\overline{\rho}_{E,p}$ be its mod p representation. Show that for all $\sigma \in G_{\mathbb{Q}}$ we have

$$\det(\overline{\rho}_{E,p}(\sigma)) = \chi_p(\sigma).$$

Hint: Think about the Weil pairing.

Exercise 4. Compute $\overline{\rho}_{E,2} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_2)$ for the following elliptic curves:

(i) $y^2 = x^3 - 3 * x^2 + 2 * x;$
(ii) $y^2 = x^3 - 2x;$
(iii) $y^2 = x^3 - 9x + 9;$
(iv) $y^2 + y = x^3 - x^2.$