Monday, Lecture 2, Ronald van Luijk, `rmluijk@gmail.com`

Please let us know about mistakes in these notes!

1. Module of Differentials

Much of this section is taken from [3], section II.8, which contains more elaborate proofs and references. See also [2], chapter 16. Although we try to define everything rigorously, the goal is to make everything explicit so we can actually do computations. Some of the proofs, however, may require more background than the mere prerequisites for this workshop.

Let A be a commutative ring (with identity) and B a commutative A-algebra.

Definition 1.1. *An A-derivation of B into a B-module M is a map $d\colon B \to M$ such that d is additive, satisfies the Leibniz rule $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$, and $d(a) = 0$ for all $a \in A$.*

Definition 1.2. *The* module of relative differential forms *of B over A is a B-module $\Omega_{B/A}$, together with an A-derivation $d\colon B \to \Omega_{B/A}$ satisfying the following universal property: for any B-module M and for any A-derivation $d'\colon B \to M$, there exists a unique B-module homomorphism $f\colon \Omega_{B/A} \to M$ such that $d' = f \circ d$.*

If $\mathrm{Der}_A(B, M)$ denotes the set of all A-derivations from B into M, then we have a natural bijection $\mathrm{Der}_A(B, M) \leftrightarrow \mathrm{Hom}_B(\Omega_{B/A}, M)$.

Proposition 1.3. *The* module of relative differential forms *$\Omega_{B/A}$ exists and is unique up to a unique isomorphism of B-modules.*

Proof. Take the free B-module generated by the symbols $\{db : b \in B\}$ and divide out by the required relations. The derivation is given by sending $b \in B$ to db. $\square$

Example 1.4. *If $B = A[x_1, \ldots, x_n]$ is a polynomial ring over A, then $\Omega_{B/A}$ is the free B-module of rank n generated by the dx_i.*

Proposition 1.5. *Let I be an ideal of B and set $C = B/I$. Then there is a natural exact sequence of C-modules*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the first map sends the image of $b \in I$ in I/I^2 to $db \otimes 1$.

Example 1.6. *If $A = k$ is a field, $B = k[x, y]$, and $C = k[x, y]/(x^2 + y^2 - 1)$, then $\Omega_{C/A}$ is the C-module generated by dx and dy with relation $2xdx + 2ydy = 0$.*

Example 1.7. *If $A = k$ is a field, and $B = k[\varepsilon]/(\varepsilon^2)$, then $\Omega_{B/A}$ is the B-module generated by $d\varepsilon$ and the relation $2\varepsilon d\varepsilon = 0$.*

There are many more interesting exact sequences describing for instance the behavior of modules of relative differential forms under tensor products, and the relation of the modules $\Omega_{B/A}$, $\Omega_{C/A}$, $\Omega_{C/B}$ for any B-algebra C, see [3], section II.8. Here we will only present what we need for our purposes.

Proposition 1.8. *If B is an integral domain with fraction field K, then $\Omega_{K/A} \cong \Omega_{B/A} \otimes_B K$.*

Let k be an algebraically closed field (don't worry, later this week we will certainly drop this assumption!). For this lecture, all our varieties will be defined over k, nonsingular, and irreducible. In particular, each variety X will have a function field, denoted by $k(X)$.

Definition 1.9. *We say that $\omega \in \Omega_{k(X)/k}$ is regular at a point $P \in X$ if there exists an affine open neighborhood $U \subset X$ of P, with coordinate ring B, such that the B-submodule $\Omega_{B/k}$ of $\Omega_{k(X)/k}$ contains ω.*

Example 1.10. *Consider $\mathbb{P}^1(x,y)$ with function field $k(t)$ for $t = x/y$, and set $\omega = dt$. Then the affine part $y \neq 0$ can be identified with $\mathbb{A}^1(t)$, which has coordinate ring $k[t]$. Since the $k[t]$-module $\Omega_{k[t]/k}$ is generated by dt, we find that ω is regular at every point of $\mathbb{A}^1$.*

Definition 1.11. *For every open subset $U \subset X$ we let $\mathcal{O}_X(U)$ denote the subring of $k(X)$ of functions that are regular at every point of U, and we let $\Omega_X(U)$ denote the $\mathcal{O}_X(U)$-submodule of $\Omega_{k(X)/k}$ consisting of differentials ω that are regular at every point of U. This defines the* structure sheaf $\mathcal{O}_X$ *and the sheaf of differentials Ω_X on the variety X.*

Proposition 1.12. *If U is an affine subvariety of X with coordinate ring B, then we have $\mathcal{O}_X(U) = \mathcal{O}_U(U) = \Omega_{B/k}$.*

The space of regular global sections is an essential invariant of a variety. For any sheaf $\mathcal{F}$ on a variety X, we will write $\Gamma(\mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

Example 1.13. *Consider $\mathbb{P}^1(x,y)$ with function field $K = k(t) = k(s)$ for $t = x/y$ and $s = t^{-1}$. Then the regular differentials on $\mathbb{A}^1(t)$ are in the $k[t]$-module generated by dt, while those on $\mathbb{A}^1(s)$ are in the $k[s]$-module generated by $ds = d(t^{-1}) = -t^{-2}dt$. The intersection in $\Omega_{K/k}$ is 0, so $\Gamma(\Omega_{\mathbb{P}^1}) = 0$, i.e., there are no nonzero differentials that are regular on $\mathbb{P}^1$.*

Exercise 1. *Consider the differential dx/y on the affine curve C in $\mathbb{A}^2(x,y)$ given by $y^2 = f(x)$ for some separable polynomial f. Show that ω is regular at every point of C. Show that this is consistent with Proposition 1.12. Show that if f has degree at least 3, then ω is in fact regular on the entire projective closure of C in $\mathbb{P}^3$.*

Exercise 2 (*)**.** *Show that if X is a hypersurface in $\mathbb{P}^n$ for $n \geq 3$, then $\Gamma(\Omega_X) = 0$.*

Exercise 3 (*)**.** *Show that if X is a complete intersection in $\mathbb{P}^n$ of dimension at least 2, then $\Gamma(\Omega_X) = 0$.*

When studying the rational points on a variety X over a number field, it is often tempting to try to map X to another variety Y on which we can control the set of points more easily. Every rational point on X would then map to a rational point of Y, so all we would need to check is which rational points of Y lift to rational points on X, a process that is particularly easy when Y does not contain any rational points. Given X, the following proposition gives restrictions on Y (to the extend that it may show this approach is useless for X).

Proposition 1.14. *If $f\colon X \to Y$ is a surjective morphism of smooth irreducible varieties, and f is generically smooth (which is automatic in characteristic 0), then then the induced map $f^*\colon \Omega_Y(Y) \to \Omega_X(X)$ is injective.*

Proof. Since f is generically smooth, there is an open subset $U \subset X$ such that $f\colon U \to Y$ is smooth (cf. [3], Lemma III.10.5). This is equivalent to saying that for any point $x \in U$, and $y = f(x)$, the induced map $T_x \to T_y$ on Zariski tangent spaces is surjective, or equivalently, the map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective, where $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote the maximal ideals of the local rings at x and y respectively (see [3], Prop. III.10.4). Now take any nonzero differential $\omega \in \Gamma(\Omega_Y)$. Since $f(U)$ is dense in Y and ω can not vanish on an open subset, there is a $y \in f(U)$ such that ω does not vanish at y. Take any $x \in U$ such that $f(x) = y$. Then by the above the map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective. These k-vectorspaces are the stalks at y and x of the sheaves $\Omega_{Y/k}$ and $\Omega_{X/k}$ respectively. Since ω does not vanish at y, its image in the stalk at y is nonzero, and by injectivity, so is its image in the stalk at x, and therefore the image of ω in $\Omega_X(X)$ is nonzero. □

Corollary 1.15. *If* $\Gamma(\Omega_X) = 0$, *and* $\operatorname{char} k = 0$, *then there is no surjective morphism from X to a nonsingular curve of positive genus or to an abelian variety of positive dimension.*

Proof. Suppose $f\colon X \to Y$ is a surjective morphism for some Y. Since the characteristic is zero, the morphism f is generically smooth, so Proposition 1.14 tells us that $\Gamma(Y, \Omega_Y) = 0$. This prevents Y from being a curve of genus $g > 0$, or an abelian variety of dimension $g > 0$, both of would satisfy $\dim_k \Gamma(\Omega_Y) = g$. □

Exercise 3.6 and Corollary 1.15 show that if X is a complete intersection of dimension at least 2, then there is no hope for a morphism from X to a curve of positive genus or an abelian variety of positive dimension.

2. Differential n-forms

For this section, we still refer to [3], II.8.

Definition 2.1. *Let K be a field extension of k. Then K is separably generated over k if there exists a transcendence base $\{t_\lambda\}$ for K/k such that K is a separable algebraic extension of $k(\{t_\lambda\})$.*

Proposition 2.2. *Let K be a finitely generated extension field of a field k. Then* $\dim_K \Omega_{K/k} \geq \operatorname{tr.d.} K/k$, *and equality holds of and only if K is separably generated over k.*

Note that when we have $K = k(X)$, then $\operatorname{tr.d.} K/k = \dim X$. Since we are assuming that X is geometrically irreducible and reduced, we find $\dim_K \Omega_{K/k} = \dim X$. If X is nonsingular, then Ω_X is a locally free shef of rank $\dim X$, which means that we can cover X by open subsets U such that for each U the $\mathcal{O}_X(U)$-module $\Omega_X(U)$ is free of rank $\dim X$.

Example 2.3. *On $\mathbb{P}^n$ we have an exact sequence of sheaves*

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Definition 2.4. *Let ω_X denote the sheaf $\bigwedge^n \Omega_X$ with $n = \dim X$. The sections of ω_X, i.e., the elements of $\omega_X(U)$, are called n-forms on X. This defines the canonical sheaf ω_X on X.*

As Ω_X is a locally free sheaf of rank $\dim X$, the canonical sheaf is a locally free sheaf of rank 1, i.e., a socalled invertible sheaf. Because of the sheafification process, it is in general not true that we get $\omega_X(U) = \bigwedge^n(\Omega_X(U))$. It is for instance possible that we have $\Gamma(\Omega_X) = 0$, while $\Gamma(\omega_X) \neq 0$. Instead of explaining sheafification, we will clarify the definition of ω_X by making it more explicit. Note that $\bigwedge^n \Omega_{k(X)/k}$ is a 1-dimensional vectorspace over $k(X)$.

Definition 2.5. *Take an n-form $\omega \in \bigwedge^n \Omega_{k(X)/k}$ and a point P on X. Let $t_1, \ldots, t_n$ be a set of local parameters at P. Then there is a unique $g \in k(X)$ such that $\omega = g dt_1 \wedge \ldots \wedge dt_n$. We say that ω is regular at P if this g is regular at P.*

Proposition 2.6. *For any open subset $U \subset X$, the $\mathcal{O}_X(U)$-module $\omega_X(U)$ consists of all n-forms $\omega \in \bigwedge^n \Omega_{k(X)/k}$ that are regular at all points $P \in U$.*

Definition 2.7. *The geometric genus of a smooth variety X over k is $g(X) = \dim_k \Gamma(X, \omega_X)$.*

Example 2.8. *For a curve C we have $\Omega_C = \omega_C$, so we have already seen that $\omega_{\mathbb{P}^1}(\mathbb{P}^1) = 0$ and thus $g(\mathbb{P}^1) = 0$.*

3. Associating a divisor to an n-form

Proposition 3.1. *The localization of a regular local ring at any prime ideal is again a regular local ring.*

Proposition 3.2. *A regular local ring of dimension 1 is a discrete valuation ring.*

Let P be a point on X, which we still assume to be nonsingular and irreducible. Let C be an irreducible closed subvariety of codimension 1 on X (i.e., a prime divisor) that contains P. Then C corresponds to a prime ideal $\mathfrak{p}$ of the local ring $\mathcal{O}_{X,P}$ of P in X. The local ring $\mathcal{O}_{X,C}$ at C in X is the localization of the regular local ring $\mathcal{O}_{X,P}$ at $\mathfrak{p}$, so it is regular as well. As $\mathcal{O}_{X,C}$ has dimension 1, it is a discrete valuation ring with associated valuation $v_C \colon k(X) \to \mathbb{Z}$.

Definition 3.3. *Take an n-form $\omega \in \bigwedge^n \Omega_{k(X)/k}$. Let $t_1, \ldots, t_n$ be a set of local parameters at P. Then there is a unique $g \in k(X)$ such that $\omega = g dt_1 \wedge \ldots \wedge dt_n$. We set $v_C(\omega) = v_C(g)$.*

Definition 3.4. *To any nonzero n-form $\omega \in \bigwedge^n \Omega_{k(X)/k}$ we associate the divisor $(\omega) = \sum_C v_C(\omega) C \in \operatorname{Div} X$, where the summation is over all prime divisors of X.*

For any two nonzero $\omega, \omega' \in \bigwedge^n \Omega_{k(X)/k}$ there is a $g \in k(X)$ such that $\omega = g\omega'$, so (ω) and (ω') are linearly equivalent.

Definition 3.5. *The class of any, and thus all, (ω) in $\operatorname{Pic} X$ is called the* canonical divisor class *of X. The divisors in this class are called* canonical divisors.

Note that if X is a projective variety in $\mathbb{P}^N$, then any two hypersurface sections of X of the same degree are linearly equivalent.

Exercise 4. *Compute the divisor (dt) on $\mathbb{P}^1(x, y)$ with $t = x/y$.*

Exercise 5. *Compute the divisor (dx/y) on the projective closure in $\mathbb{P}^2$ of the affine curve given by $y^2 = f(x)$ with f a separable polynomial of degree 2, 3, 4, general d.*

Exercise 6. *Compute the divisor $(dt_1 \wedge \ldots \wedge dt_n)$ on $\mathbb{P}^n(x_0, x_1, \ldots, x_n)$ with $t_i = x_i/x_0$.*

Exercise 7. *Let X be a hypersurface in $\mathbb{P}^n_k(x_0,\ldots,x_n)$ given by the homogeneous polynomial F of degree d, let L be any linear form in $k[x_0,\ldots,x_n]$, and set*

$$\omega = \frac{x_0^n L^{-n-1+d}}{\partial F/\partial x_0} dt_1 \wedge \ldots \wedge dt_{n-1}$$

with $t_i = x_i/x_0$. After checking that all degrees work out to make ω a well-defined element of $\bigwedge^{n-1}\Omega_{k(X)/k}$, show that we have $(\omega) = (-n-1+d)(H \cap X)$, where H is the hyperplane given by $L = 0$.

Note that with the notation of the previous exercise, there exist n-forms that are regular everywhere if and only if $d \geq n+1$, while there are no regular 1-forms if $n > 2$. The following proposition is a generalization of the previous exercise.

Proposition 3.6. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of dimension $n-k$, defined by the polynomials $F_1,\ldots,F_k$ of degrees $d_1,\ldots,d_k$ respectively. Then every canonical divisor on X is linearly equivalent to $(-n-1+\sum_{i=1}^k d_i)H$ where H is any hyperplane section of X.*

Proposition 3.6 follows from [3], Prop. II.8.20, see [3], Exerc. II.8.4. Besides the sheaf-theoretic proof given there, the following exercises also lead to a (fairly heavily) computational proof.

For any k polynomials $f_1,\ldots,f_k \in k[x_1,\ldots,x_n]$, and any sequence $J = (j_l)_{l=1}^k$ with $1 \leq j_1 < \ldots < j_k \leq n$ we define $M_J = M_J(f_1,\ldots,f_k)$ to be the determinant of the matrix $A = (\partial f_i/\partial x_{j_l})_{i,l=1}^k$.

Exercise 8. *Let $X \subset \mathbb{A}^n(x_1,)$ be a smooth complete intersection of dimension $n-k$, defined by the polynomials $f_1,\ldots,f_k \in k[x_1,\ldots,x_n]$. Let J be a sequence as above, and let I be the increasing sequence of the elements of $\{1,\ldots,n\} \setminus J$. Then up to sign the differential $\omega_J = M_J^{-1} dx_{i_1} \wedge \ldots \wedge dx_{i_{n-k}}$ is independent of the choice of J.*

Exercise 9. *Use the notation as in the previous exercise, and assume P is a point on X. Then there is a particular sequence J as in that exercise such that $M_J(P) \neq 0$ and for the corresponding sequence I, the elements $x_i - x_i(P)$ with $i \in I$ form a set of local parameters at P. Conclude that $(\omega_J) = 0$ on $X \subset \mathbb{A}^n$.*

Exercise 10. *Homogenize the previous exercises to find out the contribution to (ω) of the hyperplane at infinity of the projective closure of X. Check that your answer agrees with Proposition 3.6.*

Exercise 11. *Suppose X is a smooth complete intersection as in Proposition 3.6, and assume that X is a surface. Compute the self-intersection of a canonical divisor on X.*

The following exercise gives another generalization of exercise 7.

Exercise 12. *Let $\mathbb{P}(w_0,w_1,\ldots,w_n)$ be weighted projective n-space with coordinates $x_0,\ldots,x_n$ such that x_i has weight w_i, and assume $w_0 = 1$. Let X be a smooth hypersurface in $\mathbb{P}(w_0,w_1,\ldots,w_n)$ of (weighted) degree d. Set $D = X \cap H$ where H is the hyperplane given by $x_0 = 0$. Then any canonical divisor on X is linearly equivalent to $(d - \sum_i w_i)D$.*

Exercise 13. *Find an example of a variety X of dimension n for which the map $\bigwedge^n(\Gamma(\Omega_X)) \to \Gamma(\omega_X)$ is not surjective.*

4. Morphisms to projective space

For more details about the subject of this section, see [3], section II.7.

To any locally free sheaf $\mathcal{F}$ on X of rank 1, an open subset $U \subset X$, and a sequence $(s_0, \ldots, s_r)$ of sections of $\mathcal{F}(U)$, not all 0, we can associate a rational map from U to $\mathbb{P}^r$ by using the sections as coordinates. Even though the s_i are not necessarily functions on U, this still works because the stalk of $\mathcal{F}$ at a point $P \in U$ has dimension 1 over k, so any two elements s_i, s_j in this stalk determine a well-defined ratio, provided they do not both vanish at P.

Example 4.1. *Let D be a divisor on X and let $\mathcal{L}(D)$ be the sheaf of functions with a pole at most at D. More precisely, we have $\mathcal{L}(D)(U) = \{ f \in k(X) : (f)|_U + D|_U \geq 0 \} \cup \{0\}$. Assume $\Gamma(\mathcal{L}(D))$ is nonzero. Let $s_0, \ldots, s_r$ be a basis for $\Gamma(\mathcal{L}(D))$. In this case the s_i actually are functions and we get a rational map $X \dashrightarrow \mathbb{P}^r$ defined by $P \mapsto (s_0(P) : \ldots : s_r(P))$.*

Example 4.2. *Let X be a projective subvariety of $\mathbb{P}^n(x_0, \ldots, x_n)$. Let $H \subset \mathbb{P}^n$ be the hyperplane given by $x_0 = 0$ and set $D = X \cap H$. The functions x_i/x_0 are contained in $\Gamma(\mathcal{L}(D))$ and determine the embedding of X in $\mathbb{P}^n$. The functions $x_i x_j / x_0^2$ are contained in $\Gamma(\mathcal{L}(2D))$ and determine the embedding of X in $\mathbb{P}^N$ with $N = \binom{n+2}{2} - 1$. (Note that we did not claim to be taking a basis here.)*

Exercise 14. *Let C be the curve in $\mathbb{P}^3(x, y, z, w)$ parametrized by $(u^4 : u^3 t : u t^3 : t^4)$. Let H be the hyperplane given by $w = 0$ and set $D = C \cap H$. Show that the functions $1, x/w, y/w, z/w$ do not generate $\Gamma(\mathcal{L}(D))$. (Hint: find an isomorphism from C to $\mathbb{P}^1$ and find what divisor D corresponds to on $\mathbb{P}^1$.)*

The next proposition will be needed in an exercise. Note that any smooth variety is normal.

Proposition 4.3. *Let X be a normal complete intersection in $\mathbb{P}^n$ and H a hyperplane in $\mathbb{P}^n$. Then the map $\Gamma(\mathbb{P}^n, \mathcal{L}(nH)) \to \Gamma(X, \mathcal{L}(nH \cap X))$ is surjective.*

Proof. See [3], exerc. II.8.4(c). $\square$

Exercise 15. *Use Proposition 4.3 to show that the geometric genus of a hypersurface in $\mathbb{P}^n$ of degree d equals $\binom{d-1}{n}$.*

Example 4.4. *Let $\omega_0, \ldots, \omega_r$ be global sections of $\Gamma(\omega_X)$. Since $\bigwedge^n \Omega_{k(X)/k}$, with $n = \dim X$, is 1-dimensional over $k(X)$, there are rational functions $g_1, \ldots, g_r$ such that $\omega_i = g_i \omega_0$. The asssociated rational map is then given by $P \mapsto (1 : g_1(P) : \ldots : g_r(P))$.*

Exercise 16. *Show that the g_i in Example 4.4 are global sections of $\mathcal{L}(D)$ with $D = (\omega_0)$. More precisely, show that for any $\omega \in \bigwedge^n \Omega_{k(X)/k}$, the sheaf ω_X is isomorphic to $\mathcal{L}(D)$ for $D = (\omega)$.*

In all examples we were in fact able to find the ratios of the sections as rational functions globally, rather than finding the ratios in the stalks at the points. This reflects the fact that every invertible sheaf is in fact isomorphic to a subsheaf of the constant sheaf on X associated to $k(X)$ (see [3], Prop. II.6.15).

Definition 4.5. *We say that a divisor D on a projective variety X is very ample if a basis of sections of $\Gamma(\mathcal{L}(D))$ determines a morphism $X \to \mathbb{P}^n$ that is an immersion.*

Exercise 17. *Show that any divisor that is linearly equivalent to a very ample divisor, is in fact itself very ample.*

Example 4.6. *If X is embedded in $\mathbb{P}^n$, then any hyperplane section of X is a very ample divisor.*

Conversely, every very ample divisor on X is of this form for some embedding $X \to \mathbb{P}^r$.

Definition 4.7. *A divisor D on X is called ample is some positive multiple of D is very ample.*

Example 4.8. *Let C be a smooth curve. Then any divisor of positive degree is ample by Riemann-Roch. However, a divisor of degree 1 is very ample if and only if C has genus 0.*

Example 4.9. *Consider the cone X given by $x^2 + y^2 = z^2$ in $\mathbb{P}^3$. Show that any two lines on X through the vertex of X are linearly equivalent. Show that each of these lines is ample, but not very ample (ok, this is cheating, as we said X would always be smooth; tomorrow we will see del Pezzo surfaces, for some of which the anticanonical sheaf is ample, yet not very ample).*

Exercise 18. *Find all sequences $(d_1, \ldots, d_r)$ with $d_i \geq 2$ such that a canonical divisor on a smooth complete intersection X in $\mathbb{P}^{r+2}$ of hypersurfaces of degree $d_1, \ldots, d_r$ is not very ample. (Compare this to the next lecture.)*

References

[1] Michel Demazure. Surfaces de Del Pezzo - II. *Séminaire sur les Singularités des Surfaces*, 777:viii+339, 1980. eds. Demazure, Michel and Pinkham, Henry Charles and Teissier, Bernard, Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977.

[2] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[3] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[4] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[5] J. Milne. *Class Field Theory*. `http://www.jmilne.org/math/CourseNotes/math776.html`.

[6] B. Poonen. *Rational points on varieties*. `http://math.berkeley.edu/~poonen/math274.html`.