The Brauer–Manin obstruction

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1 Definitions

Let X be a smooth, geometrically irreducible variety over a field k. Recall that the defining property of an Azumaya algebra $\mathcal{A}$ is that, for any field K containing k and any point $P \in X(K)$, we can evaluate $\mathcal{A}$ at P to get a central simple algebra $\mathcal{A}(P)$ over K.

In particular, suppose that k is a number field. Then, for each place v of k, we can evaluate $\mathcal{A}$ at points of $X(k_v)$ to get central simple algebras over k_v. In this way we obtain a map

$$X(k_v) \xrightarrow{P \mapsto \mathcal{A}(P)} \operatorname{Br} k_v \xrightarrow{\operatorname{inv}_v} \mathbb{Q}/\mathbb{Z}. \tag{1}$$

Proposition 1.1. *The map* (1) *is continuous, and hence locally constant, for the real, complex or p-adic topology, as appropriate, on $X(k_v)$.*

Proof. I need to find a reference for this. □

Recall that the set of adelic points $X(\mathbb{A}_k)$ of X is simply the direct product $\prod_v X(k_v)$. Adding together all the local maps (1), we obtain a continuous map

$$X(\mathbb{A}_k) \to \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum_v \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z}. \tag{2}$$

By using a direct sum instead of a direct product for the middle term above, we are implicitly stating the following result:

Proposition 1.2. *Let X be a smooth, geometrically irreducible variety over a number field k, and let $\mathcal{A} \in \operatorname{Br} X$ be an Azumaya algebra on X. Then, for all but finitely many places v, we have $\mathcal{A}(P) = 0 \in \operatorname{Br} k_v$ for all $P \in X(k_v)$.*

Proof. See [3, p. 101]. □

Proposition 1.3. *If $\mathcal{A} \in \operatorname{Br} X$ lies in the image of the natural map $\operatorname{Br} k \to \operatorname{Br} X$, then the associated map* (2) *is zero.*

Proof. This is a restatement of (**??**). □

The following observation is key to the definition of the Brauer–Manin obstruction.

Proposition 1.4. *Let X be a smooth, geometrically irreducible variety over a number field k, and consider $X(k)$ as a subset of $X(\mathbb{A}_k)$ under the diagonal embedding. Let $\mathcal{A}$ be an Azumaya algebra on X. Then $X(k)$ lies in the kernel of the map* (2).

Proof. It is straightforward to check that the following diagram commutes:

$$\begin{array}{ccccc} X(k) & \longrightarrow & X(\mathbb{A}_k) & & \\ {\scriptstyle\mathcal{A}}\downarrow & & \downarrow{\scriptstyle\mathcal{A}} & & \\ \operatorname{Br} k & \longrightarrow & \bigoplus_v \operatorname{Br} k_v & \xrightarrow{\sum_v \operatorname{inv}_v} & \mathbb{Q}/\mathbb{Z} \end{array} \tag{3}$$

where the vertical arrows are evalution of $\mathcal{A}$ at points; the top horizontal arrow is the inclusion of $X(k)$ in $X(\mathbb{A}_k)$; and the bottom line is the exact sequence (**??**). The composite map from $X(\mathbb{A}_k)$ to $\mathbb{Q}/\mathbb{Z}$ is the map of (2). The proposition follows immediately from the exactness of the bottom row. □

With this in mind, we make the following definition.

Definition 1.5. Let X be a smooth, geometrically irreducible variety over a number field k, and let $\mathcal{A} \in \operatorname{Br} X$ be an Azumaya algebra on X. Define

$$X(\mathbb{A}_k)^{\mathcal{A}} := \{(P_v) \in X(\mathbb{A}_k) \mid \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0\}.$$

If B is a subset of $\operatorname{Br} X$, similarly define

$$X(\mathbb{A}_k)^{B} := \{(P_v) \in X(\mathbb{A}_k) \mid \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0 \text{ for all } \mathcal{A} \in B\}.$$

One way to look at this is as follows: the maps (2) define a pairing $X(\mathbb{A}_k) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z}$, and we have defined $X(\mathbb{A}_k)^B$ to be the subset of $X(\mathbb{A}_k)$ orthogonal to the set B under this pairing.

Remark 1.6. In view of Proposition 1.3, this pairing is actually still defined when $\operatorname{Br} X$ is replaced by $\operatorname{Br} X / \operatorname{Br} k$ (meaning the quotient of $\operatorname{Br} X$ by the image of $\operatorname{Br} k$). In many of the cases which interest us, $\operatorname{Br} X / \operatorname{Br} k$ will be a finite group, and it will be possible to calculate $X(\mathbb{A}_k)^{\operatorname{Br} X}$ explicitly.

Proposition 1.4 states that $X(k) \subseteq X(\mathbb{A}_k)^B$ for any subset B of $\operatorname{Br} X$. In particular, if $X(\mathbb{A}_k)^B$ is empty, then $X(k)$ is also empty.

Definition 1.7. Let X be a smooth, geometrically irreducible variety over a number field k. Let B be a subset of the Brauer group of X. If $X(\mathbb{A}_k)$ is not empty but $X(\mathbb{A}_k)^B$ is empty, then we say there is a *Brauer–Manin obstruction to the Hasse principle* on X coming from B. If $X(\mathbb{A}_k)^B$ is strictly contained in $X(\mathbb{A}_k)$, we say that there is a *Brauer–Manin obstruction to weak approximation* on X coming from B. If $B = \operatorname{Br} X$, we simply say that there is a Brauer–Manin obstruction to the Hasse principle or to weak approximation on X.

The reason that this is such a useful definition is that the sets $X(\mathbb{A}_k)^{\mathcal{A}}$ are often explicitly computable; for certain classes of varieties, we can even compute the set $X(\mathbb{A}_k)^{\operatorname{Br} X}$ effectively.

Proposition 1.8. *Let X be a smooth, geometrically irreducible, projective variety over a number field k, and let $\mathcal{A} \in \operatorname{Br}_1 X$. Then there is an effective procedure to compute $X(\mathbb{A}_k)^{\mathcal{A}}$.*

Proof. See [2, Section 9]. □

In some cases we can go much further:

Proposition 1.9. *Let X be a smooth, geometrically irreducible, projective variety over a number field k. Suppose that $\operatorname{Pic} \bar{X}$ is free and finitely generated, and that we are explicitly given a finite set of divisors which generate $\operatorname{Pic} \bar{X}$, together with the relations between them and the Galois action on them. Then there is an effective procedure to compute $X(\mathbb{A}_k)^{\operatorname{Br}_1 X}$.*

Proof. See [2, Theorem 3.4]. □

2 Examples

Example 2.1 (Birch–Swinnerton-Dyer [1])**.** Let X be the non-singular del Pezzo surface of degree 4 defined by the equations (**??**), and let $\mathcal{A}$ be the quaternion algebra

$$\mathcal{A} = \left(5, \frac{u}{u+v}\right)$$

over $\kappa(X)$. Then $\mathcal{A}$ is an Azumaya algebra on X, and there is a Brauer–Manin obstruction to the Hasse principle on X coming from $\mathcal{A}$.

Proof. It will be shown in the next chapter **??** that, to prove that $\mathcal{A}$ is an Azumaya algebra, it is enough to check that the principal divisor $(u/(u+v))$ is the norm of a divisor defined over $\mathbb{Q}(\sqrt{5})$. This is precisely the verification carried out for the last part of Exercise **??**.

Dividing both sides of the first equation by v^2, we see that

$$u/v = \frac{x^2 - 5y^2}{v^2} = N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}\left(\frac{x + \sqrt{5}y}{v}\right)$$

is a norm from $\mathbb{Q}(\sqrt{5})$, and so by **??** the algebra $(5, v/(u+v))$ is isomorphic to $\mathcal{A}$. In a similar way, we get the following four quaterion algebras over $\kappa(X)$, all isomorphic:

$$\mathcal{A} = \left(5, \frac{u}{u+v}\right), \qquad \left(5, \frac{v}{u+v}\right), \qquad \left(5, \frac{u}{u+2v}\right), \qquad \left(5, \frac{v}{u+2v}\right). \quad (4)$$

We will describe the map $X(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z}$, given by $P \mapsto \operatorname{inv}_v \mathcal{A}(P)$, separately for each place v.

For $v = \infty$ the real place, notice that 5 is positive and hence a square in $\mathbb{R}$; thus $\mathcal{A}(P) = (5, u(P)/(u(P)+v(P)))$ is a trivial algebra and $\operatorname{inv}_\infty \mathcal{A}(P) = 0$, at least for all $P \in X(\mathbb{R})$ where $u(P)$ and $u(P)+v(P)$ are non-zero. Since the map $P \mapsto \operatorname{inv}_\infty \mathcal{A}(P)$ is continuous on $X(\mathbb{R})$, it follows that it is zero everywhere.

If v is an odd prime p such that 5 is a square in $\mathbb{Q}_p$, then the same argument works and shows that $\operatorname{inv}_p \mathcal{A}(P) = 0$ for all $P \in X(\mathbb{Q}_p)$.

Now suppose that v is an odd prime $p \neq 5$, such that 5 is not a square in $\mathbb{Q}_p$ and therefore not a square in $\mathbb{F}_p$. In this case, u and v can never be both zero at a point of $X(\mathbb{F}_p)$, since otherwise x/y and x/z would be square roots of 5 in $\mathbb{F}_p$. Similarly, $u+v$ and $u+2v$ are never both zero at a point on $X(\mathbb{F}_p)$. It follows that, for each $P \in X(\mathbb{Q}_p)$, at least one of the isomorphic algebras (4) is clearly of the form $(5, b)$ with $b \in \mathbb{Z}_p^\times$, and therefore $\operatorname{inv}_p \mathcal{A}(P) = 0$ by Proposition **??**.

Next, consider the case $v = 2$. At first glance the previous argument will not work: since $\bar{P} = (0 : 0 : 1 : 1 : 1) \in X(\mathbb{F}_2)$, it would appear that u and v can both be even at a point of $X(\mathbb{Q}_2)$. But it turns out that $\bar{P}$ does not lift to a point of $X(\mathbb{Q}_2)$, as can be seen, for example, by looking at the equations modulo 16. So once again, for each $P \in X(\mathbb{Q}_2)$, one of u, v is odd, and similarly one of $(u+v), (u+2v)$ is odd. The formula of Proposition **??** shows that $(5, b)_2 = 1$ whenever b is odd, so once again we conclude that $\operatorname{inv}_2 \mathcal{A}(P) = 0$ for all $P \in X(\mathbb{Q}_2)$.

Finally, we look at $v = 5$. Modulo 5, the variety X reduces to a union of four planes, meeting in a common line; two of these planes are defined over $\mathbb{F}_5$ and the other two are quadratic and conjugate. The two defined over $\mathbb{F}_5$, which therefore contain all the points of $X(\mathbb{F}_5)$, are $\{u = v = x\}$ and $\{u = v = -x\}$. The line of intersection of these planes is $\{u = v = x = 0\}$, but no point here lifts to a point of $X(\mathbb{Q}_5)$. Therefore every point of $X(\mathbb{Q}_5)$ satisfies $u \equiv v \equiv \pm x \pmod 5$ with $u, v, x \in \mathbb{Z}_5^\times$. This means that $u/(u+v) \equiv 3 \pmod 5$, and the formula of Proposition **??** gives $(5, 3)_5 = -1$, meaning that $\operatorname{inv}_5 \mathcal{A}(P) = \frac{1}{2}$ for all $P \in X(\mathbb{Q}_5)$.

To summarise, we have proved that $\operatorname{inv}_v \mathcal{A}(P) = 0$ for all $P \in X(\mathbb{Q}_v)$ where $v \neq 5$, and that $\operatorname{inv}_v \mathcal{A}(P) = \frac{1}{2}$ for all $P \in X(\mathbb{Q}_5)$. It follows that

$$\sum_v \operatorname{inv}_v \mathcal{A}(P_v) = \frac{1}{2} \text{ for all } (P_v) \in X(\mathbb{A}_\mathbb{Q}) = \prod_v X(\mathbb{Q}_v).$$

So $X(\mathbb{A}_\mathbb{Q})^\mathcal{A} = \emptyset$, and therefore there is a Brauer–Manin obstruction to the Hasse principle on X. □

Remark 2.2. Although our representative quaternion algebras (4) all had problems when $u = v = 0$, it turned out that we never needed to evaluate them at such a point. In fact we could have found other isomorphic quaternion algebras, extending our set (4) to a set which could be easily evaluated at *any* point of any $X(\mathbb{Q}_v)$; this is a consequence of the fact that $\mathcal{A}$ is Azumaya.

We conclude by showing that the phenomenon of Example **??** can also be explained by the Brauer–Manin obstruction.

Example 2.3. Let X be the singular cubic surface (**??**), and let $\mathcal{A}$ be the quaternion algebra over $\kappa(X)$ defined by

$$\mathcal{A} = \left(-1, \frac{4z - 7t}{t}\right).$$

Then $\mathcal{A}$ is an Azumaya algebra on X, and there is a Brauer–Manin obstruction to weak approximation on X coming from $\mathcal{A}$, which explains why one connected component of $X(\mathbb{R})$ contains no rational points.

Remark 2.4. We have not defined what an Azumaya algebra is on a singular variety. In this particular case, it is straightforward to check that $\mathcal{A}$ is an Azumaya algebra away from the singular points of X, and this is enough for our purposes.

Proof. The two singular points of X are $(\pm i : 1 : 0 : 0)$ where $i^2 = -1$. Let U denote the complement in X of these two points. Any rational points of X must be contained in U, since neither of the singular points are rational.

As before, we start by finding alternative ways of writing the quaternion algebra $\mathcal{A}$. Firstly, note that by looking at the defining equations of X we can see that

$$\frac{4z - 7t}{t} = \frac{x^2 + y^2 + 8zt - 14t^2}{z^2}$$

as functions on X, immediately giving a new way of writing $\mathcal{A}$. Furthermore, since the denominator z^2 is a square, we can replace it with any other square such as x^2, y^2 or t^2 to get new quaternion algebras isomorphic to $\mathcal{A}$. Also, the defining equations show easily that the algebra $(-1, (z^2 - 2t^2)/t^2)$ is isomorphic to $\mathcal{A}$, and again the denominator here may be replaced by any square. In this way we find a set of isomorphic algebras which can be evaluated at any point of U.

Now consider each place separately. If $p \equiv 1 \pmod 4$ then -1 is a square in $\mathbb{Q}_p$, and so $\operatorname{inv}_p \mathcal{A}(P) = 0$ for all $P \in U(\mathbb{Q}_p)$. If $p \equiv 3 \pmod 4$ then a similar argument to the previous example shows that, for each $P \in X(\mathbb{Q}_p)$, one of our algebras always evaluates to $(-1, b)$ with $b \in \mathbb{Z}_p^\times$, and therefore that $\operatorname{inv}_p \mathcal{A}(P) = 0$ for all $P \in X(\mathbb{Q}_p)$.

For $v = 2$, notice that $(4z - 7t)/t \equiv 1 \pmod 4$, and that $(-1, b)_2 = 1$ whenever $b \equiv 1 \pmod 4$, so again $\operatorname{inv}_2 \mathcal{A}(P) = 0$ for all $P \in X(\mathbb{Q}_2)$.

Finally, for $v = \infty$, we see that $(4z - 7t)/t$ is non-negative on one component of $X(\mathbb{R})$, and strictly positive on a dense open subset of that component, so $\operatorname{inv}_\infty \mathcal{A} = 0$ on that component. On the other component, however, $(4z - 7t)/t$ is strictly negative on a dense open subset, and therefore $\operatorname{inv}_\infty \mathcal{A} = \frac{1}{2}$ on that component. We deduce that rational points can only be found in the component where $(4z - 7t)/t \geq 0$, giving a Brauer–Manin obstruction to weak approximation on X. □

References

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